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Abstract: We introduce a three-dimensional Hybrid High-Order method for magnetostatic problems. The proposed method is easy to implement, supports general polyhedral meshes, and allows for arbitrary orders of approximation.

Key words: Hybrid High-Order methods, polyhedral meshes, nonconforming methods, magnetostatics

MSC (2010)**:** 65N08, 65N12, 65N30

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded and connected polyhedral domain, with boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. We assume that Ω is topologically trivial, and that $\partial\Omega$ is connected. For any $X \subset \overline{\Omega}$, we denote by $(\cdot,\cdot)_X$ and $||\cdot||_X$ the usual inner product and norm on $L^2(X,\mathbb{R}^l)$, $l \in \{1,2,3\}$. The standard magnetostatic problem consists in finding the magnetic field $\mathbf{u} : \Omega \to \mathbb{R}^3$ such that

$$\mathbf{curl}\,\mathbf{u} = \mathbf{f} \qquad \text{in } \Omega, \tag{1a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$\mathbf{n}\times\mathbf{u}\times\mathbf{n} = \mathbf{0} \qquad \text{on } \partial\Omega, \tag{1c}$$

where $\mathbf{H}(\mathrm{div};\Omega) \ni \mathbf{f} : \Omega \to \mathbb{R}^3$ denotes the current density and is such that $\operatorname{div}\mathbf{f} = 0$ in Ω and $\mathbf{f}\cdot\mathbf{n} = 0$ on $\partial\Omega$. We supplement Problem (1) with another unknown, namely

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the potential $p : \Omega \to \mathbb{R}$, that satisfies $p = 0$ in Ω. From now on, Problem (1) refers to this augmented problem with unknowns $(\mathbf{u}, p)$. The starting point of our discretization is the following equivalent weak formulation of Problem (1), originally introduced in [8, Eq. (58)]: Find $(\mathbf{u}, p) \in \mathbf{X_0} \times Y_0$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{curl}\,\mathbf{v})_\Omega \qquad \forall \mathbf{v} \in \mathbf{X_0}, \tag{2a}$$
$$-b(\mathbf{u}, q) + c(p, q) = 0 \qquad \forall q \in Y_0, \tag{2b}$$

where

$$\mathbf{X_0} := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) \ : \ \mathbf{n} \times \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}, \qquad Y_0 := H^1_0(\Omega),$$

and the bilinear forms $a : \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathbf{curl}; \Omega) \to \mathbb{R}$, $b : \mathbf{H}(\mathbf{curl}; \Omega) \times H^1(\Omega) \to \mathbb{R}$, and $c : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ are given by

$$a(\mathbf{w},\mathbf{v}) := (\mathbf{curl}\,\mathbf{w}, \mathbf{curl}\,\mathbf{v})_\Omega, \qquad b(\mathbf{w}, q) := (\mathbf{w}, \nabla q)_\Omega, \qquad c(r, q) := (r, q)_\Omega. \tag{3}$$

Testing (2a) with $\mathbf{v} = \nabla p \in \mathbf{X_0}$, it is inferred that $p = 0$ in Ω. The well-posedness of Problem (2) is then a consequence of the coercivity of a on the subspace of $\mathbf{X_0}$ given by $\{\mathbf{w} \in \mathbf{X_0} \ : \ b(\mathbf{w}, q) = 0 \quad \forall q \in Y_0\} = \{\mathbf{w} \in \mathbf{X_0} \ : \ \operatorname{div} \mathbf{w} = 0\}$ which, in turn, follows from the first Weber inequality (see, e.g., [1, Theorem 3.4.3]).

Various discretization methods have been studied in the literature to approximate the Maxwell equations. We can, in particular, cite the seminal work of [9] on simplicial elements. On more general element shapes, one can mention the Discontinuous Galerkin method of [11], the Hybridizable Discontinuous Galerkin (HDG) methods of [10] and [3], or the Virtual Element method of [12].

In this paper, we devise an easy-to-implement Hybrid High-Order (HHO) method to solve Problem (1). HHO methods have been originally introduced in [7, 6]. Their connections with HDG methods have been later discussed in [4] in the context of scalar variable diffusion problems. The method we introduce here shares some similarities with the HDG method of [3]. It indeed hinges, as in [3], on face unknowns for the magnetic field belonging to a subtle subspace of $\mathbb{P}^{k+1}(F; \mathbb{R}^2)$. However, there are two main differences between our method and the one in [3]. First, taking advantage of the fact that Problem (1) is actually first-order, we do not (locally) reconstruct a discrete **curl** operator. We hence (i) can consider a smaller local set of face unknowns, and (ii) we do not have to solve a local problem on each mesh cell (which may become, for a sequential implementation, rather costly in 3D, especially for large polynomial degrees). Second, and as opposed to [3] in which the bilinear form c is not introduced (therein, p may be nonzero and the authors are also interested in its approximation, which is not our case), we consider the formulation (2) of Problem (1). At the discrete level, it enables to improve the stability of the method without jeopardizing the approximation of $\mathbf{u}$.

The rest of the paper is organized as follows. In Section 2 we describe the discrete setting and our HHO discretization. In Section 3 we state the discrete problem and discuss its well-posedness. Finally, in Section 4, we numerically validate the proposed method.

2 Hybrid High-Order discretization

2.1 Discrete setting

We consider sequences of refined meshes that are admissible in the sense of [5, Definition 1.9]. Each mesh $\mathscr{T}_h$ in the sequence is a finite collection $\{T\}$ of nonempty, disjoint, open polyhedra that are assumed to be star-shaped with respect to some interior point. There holds $\overline{\Omega} = \bigcup_{T\in\mathscr{T}_h} \overline{T}$ with $h = \max_{T\in\mathscr{T}_h} h_T$, where h_T denotes the diameter of the cell T. For all $T \in \mathscr{T}_h$, the boundary of T is decomposed into planar faces collected in the set $\mathscr{F}_T$. For admissible mesh sequences, card($\mathscr{F}_T$) is bounded uniformly in h. Interfaces are collected in the set $\mathscr{F}_h^{\mathrm{i}}$, boundary faces in the set $\mathscr{F}_h^{\mathrm{b}}$, and we define $\mathscr{F}_h := \mathscr{F}_h^{\mathrm{i}} \cup \mathscr{F}_h^{\mathrm{b}}$. For all $T \in \mathscr{T}_h$ and all $F \in \mathscr{F}_T$, the diameter of F is denoted h_F and the unit normal to F pointing outward T is denoted $\mathbf{n}_{TF}$. For admissible mesh sequences, h_F is uniformly comparable to h_T.

2.2 Discrete unknowns

Let an arbitrary polynomial degree $k \geq 0$ be given. For $X \in \{F,T\}$ and, respectively, $d \in \{2,3\}$, and for $l \in \{1,2,3\}$, we denote by $\mathbb{P}^k(X,\mathbb{R}^l)$ the vector space of d-variate, l-valued polynomial functions on X of total degree at most k. When $l = 1$, we simply write $\mathbb{P}^k(X)$. The global sets of discrete unknowns for the magnetic field and the potential are given by

$$\underline{\mathbf{X}}_h^{k+1} := \left\{ \underline{\mathbf{v}}_h = \big((\mathbf{v}_T)_{T\in\mathscr{T}_h}, (\mathbf{v}_F)_{F\in\mathscr{F}_h}\big) \ : \ \begin{array}{ll} \mathbf{v}_T \in \mathbb{P}^{k+1}(T;\mathbb{R}^3) & \forall T \in \mathscr{T}_h \\ \mathbf{v}_F \in \nabla_\tau \mathbb{P}^{k+2}(F) & \forall F \in \mathscr{F}_h \end{array} \right\},$$
$$\underline{\mathrm{Y}}_h^{k+1} := \left\{ \underline{\mathrm{q}}_h = \big((\mathrm{q}_T)_{T\in\mathscr{T}_h}, (\mathrm{q}_F)_{F\in\mathscr{F}_h}\big) \ : \ \begin{array}{ll} \mathrm{q}_T \in \mathbb{P}^{k}(T) & \forall T \in \mathscr{T}_h \\ \mathrm{q}_F \in \mathbb{P}^{k+1}(F) & \forall F \in \mathscr{F}_h \end{array} \right\},$$

where, for all $F \in \mathscr{F}_h$, $\nabla_\tau \mathbb{P}^{k+2}(F)$ denotes the space of (tangential) gradients of polynomials of degree $k+2$ on F. For all $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_h^{k+1}$, $\mathbf{v}_h$ (not underlined) denotes the function in the broken space $\mathbb{P}^{k+1}(\mathscr{T}_h;\mathbb{R}^3)$ such that $\mathbf{v}_{h|T} := \mathbf{v}_T$ for all $T \in \mathscr{T}_h$.

Remark 1. In [3], the authors consider face unknowns $\mathbf{v}_F$ in the larger space

$$\mathbb{P}^k(F;\mathbb{R}^2) \oplus \nabla_\tau \widetilde{\mathbb{P}}^{k+2}(F),$$

where $\widetilde{\mathbb{P}}^{k+2}(F)$ is the space of homogeneous polynomials of degree $k+2$ on F.

We define the interpolators $\underline{\mathbf{I}}_{\mathbf{X},h}^{k+1} : H^1(\Omega;\mathbb{R}^3) \to \underline{\mathbf{X}}_h^{k+1}$ and $\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1} : H^1(\Omega) \to \underline{\mathrm{Y}}_h^{k+1}$ such that, for any $\mathbf{v} \in H^1(\Omega;\mathbb{R}^3)$ and $q \in H^1(\Omega)$,

$$\underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\mathbf{v} := \Big(\big(\pi_T^{k+1}(\mathbf{v}_{|T})\big)_{T\in\mathscr{T}_h}, \big(\pi_F^{k+1,\nabla}\gamma_\tau(\mathbf{v}_{|F})\big)_{F\in\mathscr{F}_h}\Big), \tag{4a}$$

$$\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1} q := \Big(\big(\pi_T^{k}(q_{|T})\big)_{T\in\mathscr{T}_h}, \big(\pi_F^{k+1}(q_{|F})\big)_{F\in\mathscr{F}_h}\Big), \tag{4b}$$

where (i) $\gamma_\tau(\mathbf{v}_{|F}) \in L^2(F;\mathbb{R}^2)$ denotes the tangential trace of $\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)$ on F, (ii) for $X\in\{F,T\}$ and $q\in\mathbb{N}$, π_X^q denotes, for $l\in\{1,2,3\}$, the $L^2(X;\mathbb{R}^l)$-orthogonal projector onto $\mathbb{P}^q(X;\mathbb{R}^l)$, and (iii) $\pi_F^{k+1,\nabla}$ denotes the $L^2(F;\mathbb{R}^2)$-orthogonal projector onto $\nabla_\tau\mathbb{P}^{k+2}(F)$. In what follows, we denote by π_h^q the global L^2-orthogonal projector such that, for all $T\in\mathscr{T}_h$, $\pi_{h|T}^q := \pi_T^q$.

We finally introduce the following global sets of discrete unknowns, that enforce the zero Dirichlet boundary conditions:

$$\underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} := \Big\{ \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_h^{k+1} \;:\; \mathbf{v}_F \equiv \mathbf{0}\ \forall F\in\mathscr{F}_h^{\mathrm{b}} \Big\},$$
$$\underline{\mathrm{Y}}_{h,0}^{k+1} := \Big\{ \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_h^{k+1} \;:\; \mathrm{q}_F \equiv 0\ \forall F\in\mathscr{F}_h^{\mathrm{b}} \Big\}.$$

2.3 Discrete bilinear forms

The discrete counterpart of the bilinear form a defined in (3) is the bilinear form $\mathrm{a}_h : \underline{\mathbf{X}}_h^{k+1}\times\underline{\mathbf{X}}_h^{k+1}\to\mathbb{R}$ given by

$$\mathrm{a}_h(\underline{\mathbf{w}}_h,\underline{\mathbf{v}}_h) := (\mathbf{curl}_h\,\mathbf{w}_h, \mathbf{curl}_h\,\mathbf{v}_h)_\Omega + \mathrm{s}_h(\underline{\mathbf{w}}_h,\underline{\mathbf{v}}_h), \tag{5}$$

where $\mathbf{curl}_h$ denotes the broken $\mathbf{curl}$ operator on $\mathscr{T}_h$ and $\mathrm{s}_h : \underline{\mathbf{X}}_h^{k+1}\times\underline{\mathbf{X}}_h^{k+1}\to\mathbb{R}$ is the stabilization bilinear form such that

$$\mathrm{s}_h(\underline{\mathbf{w}}_h,\underline{\mathbf{v}}_h) := \sum_{T\in\mathscr{T}_h}\sum_{F\in\mathscr{F}_T} h_F^{-1}\big(\pi_F^{k+1,\nabla}(\mathbf{w}_F-\gamma_\tau(\mathbf{w}_{T|F})), \pi_F^{k+1,\nabla}(\mathbf{v}_F-\gamma_\tau(\mathbf{v}_{T|F}))\big)_F.$$

On the other hand, the discrete coupling bilinear form $\mathrm{b}_h : \underline{\mathbf{X}}_h^{k+1}\times\underline{\mathrm{Y}}_h^{k+1}\to\mathbb{R}$ is given by

$$\mathrm{b}_h(\underline{\mathbf{w}}_h,\underline{\mathrm{q}}_h) := \sum_{T\in\mathscr{T}_h}\left(-(\mathrm{q}_T, \operatorname{div}\mathbf{w}_T)_T + \sum_{F\in\mathscr{F}_T}\big(\mathrm{q}_F, \mathbf{w}_{T|F}\cdot\mathbf{n}_{TF}\big)_F\right). \tag{6}$$

From the definitions (6) and (4b) of, respectively, b_h and $\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1}$, one can easily prove the following commutation property: For any $q\in H^1(\Omega)$,

$$\mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1} q) = (\mathbf{w}_h, \nabla q)_\Omega \qquad \text{for all } \underline{\mathbf{w}}_h\in\underline{\mathbf{X}}_h^{k+1}. \tag{7}$$

Finally, the discrete counterpart of the bilinear form c is the bilinear form $\mathrm{c}_h : \underline{\mathrm{Y}}_h^{k+1}\times\underline{\mathrm{Y}}_h^{k+1}\to\mathbb{R}$ given by

$$\mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) := \sum_{T \in \mathscr{T}_h} \left((\mathrm{r}_T, \mathrm{q}_T)_T + \sum_{F \in \mathscr{F}_T} h_F (\mathrm{r}_F, \mathrm{q}_F)_F \right).$$

One can easily see that $\mathrm{c}_h(\cdot,\cdot)^{1/2}$ defines a norm on $\underline{\mathrm{Y}}_h^{k+1}$.

3 Discrete problem

Our HHO discretization of Problem (2) reads: Find $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ such that

$$\mathrm{a}_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + \mathrm{b}_h(\underline{\mathbf{v}}_h, \underline{\mathrm{p}}_h) = (\mathbf{f}, \mathbf{curl}_h \, \mathbf{v}_h)_\Omega \qquad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1}, \tag{8a}$$

$$-\mathrm{b}_h(\underline{\mathbf{u}}_h, \underline{\mathrm{q}}_h) + \mathrm{c}_h(\underline{\mathrm{p}}_h, \underline{\mathrm{q}}_h) = 0 \qquad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}. \tag{8b}$$

Some remarks are in order.

Remark 2 (Well-posedness). At the discrete level, $\underline{\mathrm{p}}_h$ is a priori nonzero. The well-posedness of Problem (8) hinges on the following discrete Weber inequality, whose proof will be given in the forthcoming article [2]: For all $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ such that

$$-\mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) + \mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) = 0 \qquad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}, \tag{9}$$

it holds

$$\|\mathbf{w}_h\|_\Omega^2 \lesssim \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h) + \mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h). \tag{10}$$

Note that the commutation property (7) is instrumental to prove (10). Remark, as well, that the discrete solution $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)$ to Problem (8) satisfies (9). The inequality (10) implies that $|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)|_{\mathrm{e},h}^2 := \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h) + \mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h)$ defines a norm on the subspace of $\underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ given by (9). As a consequence, the bilinear form of Problem (8), that is

$$\mathscr{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) := \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) + \mathrm{b}_h(\underline{\mathbf{v}}_h, \underline{\mathrm{r}}_h) - \mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) + \mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h),$$

is coercive on the latter subspace. This is not true if c_h is only a semi-norm on $\underline{\mathrm{Y}}_{h,0}^{k+1}$, as is the case in [3].

Remark 3 (Algebraic aspects). We point out that all the element unknowns can be locally eliminated, resulting in a global system written in terms of face unknowns only. In Table 1 we collect the dimensions, for several values of k, of the local sets of face unknowns for both the potential and the magnetic field, and we provide a comparison with [3].

Remark 4 (Convergence rates). For smooth enough solutions, the error in discrete energy-norm $|\cdot|_{\mathrm{e},h}$ is expected to be of order $k+1$, whereas an order $k+2$ is expected for the L^2-error on the magnetic field. Details will be given in [2]. Recall that we are not interested here in the approximaton of $p = 0$, but only in that of $\mathbf{u}$.

k	$\dim\left[\mathbb{P}^{k+1}(F)\right]$	$\dim\left[\nabla_\tau \mathbb{P}^{k+2}(F)\right]$	$\dim\left[\mathbb{P}^{k+1}(F)\right]$	$\dim\left[\mathbb{P}^{k}(F;\mathbb{R}^2) \oplus \nabla_\tau \widetilde{\mathbb{P}}^{k+2}(F)\right]$
0	3	5	3	5
1	6	9	6	10
2	10	14	10	17
3	15	20	15	26

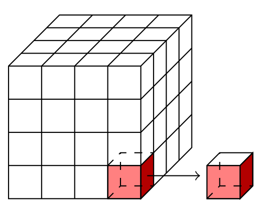
Table 1 Dimensions of the local sets of face unknowns for the potential (first column) and the magnetic field (second column), for both our method (left) and the one of [3] (right).

4 Numerical experiments

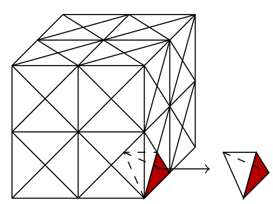
We let Ω be the unit cube, and we consider the following smooth solution:

$$\mathbf{u}(x_1,x_2,x_3) \coloneqq \begin{pmatrix} \sin(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}. \tag{11}$$

One can easily verify that $\mathbf{u}$ defined by (11) satisfies (1b) and the boundary condition (1c). The expression of the source term $\mathbf{f}$ is inferred from (1a). The numerical experiments are performed on two mesh families, a cubic one and a regular tetrahedral one, as shown on Figure 1. Element unknowns are locally eliminated, and the resulting (condensed) global linear system is solved using the SparseLU direct solver of the Eigen library, on an Intel Xeon E5-2680 v4 2.4 GHz with 128 Go of RAM. We display on Figures 2 and 3 the relative errors as functions of, respectively, the meshsize, the solution time in seconds, i.e. the time needed to solve the (condensed) global linear system, and the number of (interface) degrees of freedom (DoF). For both mesh families, the observed convergence orders are, as expected, (i) $k+1$ for the error $\mathrm{a}_h(\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\mathbf{u}, \underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\mathbf{u})^{1/2}$, and (ii) $k+2$ for the error $\|\mathbf{u}_h - \pi_h^{k+1}\mathbf{u}\|_\Omega$. Figures 2 and 3 also clearly exemplify the fact that, whenever the solution is smooth enough (at least locally), if one wants to increase the accuracy, then raising the polynomial degree is computationally much more efficient than refining the mesh.



(a) Cubic



(b) Regular tetrahedral

Fig. 1 Mesh families for the numerical tests.

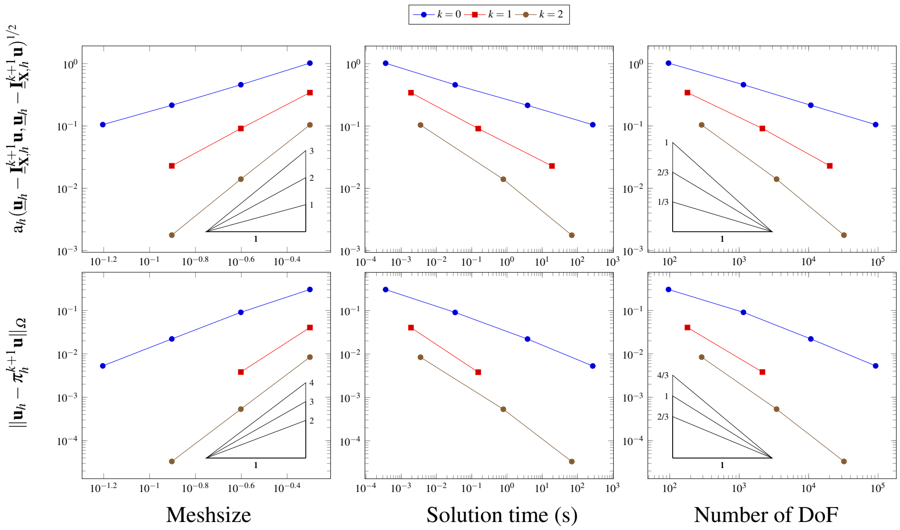


Fig. 2 Errors vs. h (left column), solution time (middle column), and number of DoF (right column) on cubic meshes.

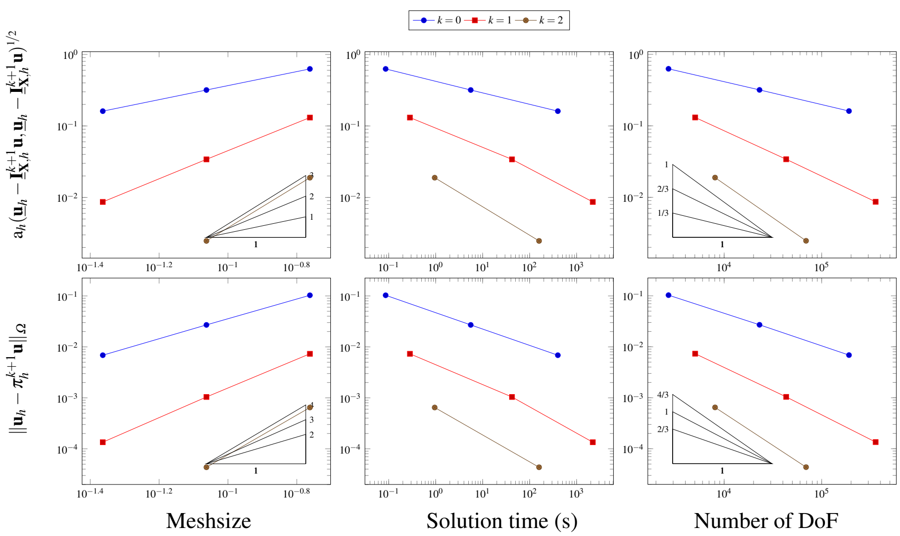


Fig. 3 Errors vs. h (left column), solution time (middle column), and number of DoF (right column) on regular tetrahedral meshes.

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